数理解析研究所講究録
237巻 1975年 51-54

RESUME

(A Note on Isolated Singularity)

Isao NARUKI

0. Notation

(X,x): pair of analytic space and $x \in X$ such that $X \setminus x$ smooth.

ι: $X \setminus x \hookrightarrow X$ (inclusion).

Ω_X^p: sheaf of (analytic) p-forms on X ($\mathcal{O}_X = \Omega_X^0$).

1. Duality

Lemma 1. $R^q \iota_* \iota^* \Omega_X^p \xleftrightarrow{\text{dual}} R^{n-q-1} \iota_* \iota^* \Omega_X^{n-p}$.

$(1 \leq q \leq n-2,\ n = \dim X)$.

Proof. Andreotti-Grauert [1].

2. Coherence of Local Cohomology

Lemma 2. S: coherent $\mathcal{O}_X$-Modele, $S|X \setminus x$: locally free

$\Rightarrow \mathcal{H}_x^q(S)$: coherent for $q < \dim X$

Proof. Siu [7].

3. Condition (L), Poincaré Lemma

Definition. (X,x) satisfies $(L) \Leftrightarrow \mathcal{H}_x^q(\Omega_X^p) = 0$ if $p+q < \dim X$.

Lemma 3. (Partial Poincaré Lemma). (X,x) satisfies (L)

$\Rightarrow H^p(\Omega_{X,x}^{\cdot}) = 0 \quad (0 < p < \dim X)$.

Proof. Bloom-Herrera [2].

4. Nice Function, Hypersurfacesection

Definition. $f\colon X \xrightarrow{\text{analytic}} \mathbb{C}$ is nice on (X,x)

$\iff f(x)=0,\ \ df_z \neq 0 \ \ (\forall z \in X\backslash x)$.

(Y,y): hypersurfacesection of (X,x)

$\iff \exists f$: nice on (X,x) such that $(Y,y) \overset{\text{iso.}}{\simeq} (f^{-1}(0),x)$.

Lemma 4 (de Rham Lemma). f: nice on (X,x), (X,x) satisfies

(L) $\Rightarrow\ 0 \to \Omega_X^0 \xrightarrow{df} \Omega_X^1 \to \cdots \to \Omega_X^{\dim X}$ exact.

5. Conservation of (L).

Let (Y,y): hypersurface section of (X,x)

Theorem 1. (X,x) satisfies (L)

$$\iff \begin{cases} \text{i) } (Y,y) \text{ satisfies (L)} \\ \text{ii) } \dim \mathcal{H}_y^0(\Omega_Y^n) = \dim \mathcal{H}_y^1(\Omega_Y^n) \end{cases}$$

$(n=\dim Y \geq 2)$

Corollary. (X,x): a complete intersection of hypersurfaces

$\Rightarrow (X,x)$ satisfies (L)

Proof. Hamm [4].

6. Milnor Fibering

Let f: nice on (X,x), (Y,y): hypersurface section defined by f. We can assume by Milnor [6]

a) $(Y,y) \hookrightarrow (X,x) \xhookrightarrow{\text{closed}} (B,0)$ (B: open ball in $\mathbb{C}^N\colon (z_1, z_2, \cdots z_N)$)

b) $r|X\backslash x, r|Y\backslash y$ have no critical point $(r(z) = \sum_{i=1}^{N} |z_i|^2)$

Theorem 2. Under the above assumption, $\exists S$: a neighborhood of 0 in $\mathbb{C}$ such that i) $\mathbb{R}^p f_*(\Omega_f^\cdot)|S$ are coherent $\mathcal{O}_S$-Modules

ii) $H^p(\Omega_{f,x}^\cdot) \cong \mathbb{R}^p f_*(\Omega_f^\cdot)_0$

where we have set $\Omega_f^p = \Omega_X^p / df \wedge \Omega_X^{p-1}$ $(p=0,1,2,\cdots)$.

Proof. Brieskorn [3] and Kiehl-Verdier [5].

7. Main Theorem

Theorem 3. (X,x), f, (Y,y), $\Omega_f^\cdot$ as in §6, (X,x) satisfies (L)

$\Rightarrow$

i) $H^p(\Omega_{f,x}^\cdot) = 0 \qquad 1 \leq p \leq n-1 \qquad (n = \dim Y)$.

ii) $H^n(\Omega_{f,x}^\cdot)$: free $\mathcal{O}_{\mathbb{C},0}$-module of finite rank.

iii) $\mu = \operatorname{rank}_{\mathcal{O}_{\mathbb{C},0}} H^n(\Omega_{f,x}^\cdot) = \dim_{\mathbb{C}} H^n({}'\Omega_{Y,y}^\cdot)$

$({}'\Omega_Y^\cdot = 0 \to \Omega_Y^0 \xrightarrow{d} \Omega_Y^1 \xrightarrow{d} \cdots \xrightarrow{d} \Omega_Y^n \longrightarrow 0)$.

iv) $\mu = \dim R^1 \iota_* \iota^* \Omega_Y^{n-1} + \dim H^n(\iota_* \iota^* \Omega_Y^\cdot) - \dim H^{n-1}(\iota_* \iota^* \Omega_p^\cdot)$

v) In case (X,x) is smooth, there are isomorphisms i

$$\mathcal{H}_y^0(\Omega_Y^{n+1}) \overset{i}{\simeq} \mathcal{H}_y^1(\Omega_Y^n) \overset{i}{\simeq} \cdots\cdots \overset{i}{\simeq} \mathcal{H}_y^{n-1}(\Omega_Y^2)$$

$$\mathcal{H}_y^1(\Omega_Y^{n-1}) \overset{i}{\simeq} \mathcal{H}_y^2(\Omega_Y^{n-2}) \overset{i}{\simeq} \cdots\cdots \overset{i}{\simeq} \mathcal{H}_y^{n-1}(\Omega_Y^1).$$

Reference

[1] Andreotti-Grauert: Théorèmes de finitude pour la cohomologie des espaces complexes, Bull. Soc. Math. France 90(1972), 193-259.

[2] Bloom-Herrera: De Rham cohomology of an analytic space, Inventiones Math. 7(1969), 275-296.

[3] Brieskorn: Die Monodromie der isolierten Singularitäten von Hyperflächen, Manuscripta Math. 2(1970), 103-161.

[4] Hamm: Locale topologische Eigenschaften komplexer Räume, Math. Ann. 191(1971), 235-252.

[5] Kiehl-Verdier: Ein einfacher Beweis des Kohärenzsatzes von Grauert, Math. Ann. 195(1971).

[6] Milnor: Singular points of complex hypersurfaces, Ann. Math. Studies, No.51, Princeton Univ.

[7] Siu: Analytic sheaves of local cohomology, Trans. Amer. Math. Soc. 148(1970), 347-366.